Exercises for Chapter 2

B1. If $\boldsymbol{u}$ and $\boldsymbol{v}$ are vectors, describe geometrically the points $2\boldsymbol{u}$, $(\boldsymbol{u}+\boldsymbol{v})/2$, and $2\boldsymbol{v}-\boldsymbol{u}$.

B2. Find the solution set to $F\boldsymbol{x}=\mathbf{0}$ for

$$F = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 0 & 3 & 0 \end{bmatrix}$$

B3. Find the solution set in parametric vector form for the following system of equations:

$$\begin{aligned} a + 2b - 3c + 4d &= 8 \\ 2a \qquad + 6c \qquad &= 12 \end{aligned}$$

B4. Give, in parametric vector form, the solution set to the following linear system, with variables x_1, x_2, x_3, x_4:

$$\left[\begin{array}{cccc|c} 2 & 4 & 4 & 6 & 5 \\ 4 & 8 & 9 & 10 & 8 \\ 4 & 8 & 8 & 12 & 10 \end{array}\right]$$

B5. (a) On the same axes draw the two vectors $\boldsymbol{u}=(2,3)$ and $\boldsymbol{v}=(4,-1)$, as well as their sum and difference.

(b) What is the area of the parallelogram with sides $\boldsymbol{u}$ and $\boldsymbol{v}$?

(c) Prove that: for every pair of vectors with integer coordinates, the area of the parallelogram they describe is an integer.

B6. Wattana started with the (un-augmented) matrix C below, and using elementary row operations reached the matrix D shown.

$$C = \begin{bmatrix} 1 & 2 & 3 & -4 \\ 2 & 5 & 7 & 3 \\ 3 & 4 & 2 & 1 \\ 4 & 10 & 14 & 6 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 & 0 & -19 \\ 0 & 1 & 0 & 18 \\ 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Is the span of the columns of C all of $\mathbb{R}^4$? Justify your answer.

(b) Give two specific solutions to the matrix equation $C\boldsymbol{x}=\mathbf{0}$.

B7. Consider a quartet $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4\}$ of vectors in $\mathbb{R}^4$. For each of the following claims, state whether it is always true or sometimes false. If always true, give a justification. If sometimes false, give a counterexample.

(a) If $\boldsymbol{v}_4$ is not a linear combination of $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, and $\boldsymbol{v}_3$, then the quartet is linearly independent.

(b) If $\boldsymbol{v}_4$ is a linear combination of $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, and $\boldsymbol{v}_3$, then the quartet is linearly independent.

(c) If the quartet is linearly independent, then so is the set containing just the single vector $\boldsymbol{v}_1+\boldsymbol{v}_2+\boldsymbol{v}_3+\boldsymbol{v}_4$.

(d) If the quartet is linearly dependent, then so is the set consisting of $\boldsymbol{v}_1$, $2\boldsymbol{v}_2$, $3\boldsymbol{v}_3$, and $4\boldsymbol{v}_4$.

Some Solutions

B1. If these are vectors from the origin, then the endpoint of $2\boldsymbol{u}$ is double that of $\boldsymbol{u}$; the endpoint of $(\boldsymbol{u}+\boldsymbol{v})/2$ is halfway between that of $\boldsymbol{u}$ and $\boldsymbol{v}$; and the endpoint of $2\boldsymbol{v}-\boldsymbol{u}$ is such that the endpoint of $\boldsymbol{v}$ is halfway between it and the endpoint of $\boldsymbol{u}$.

B2. All multiples of

$$\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

B3.

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} -3 \\ 3 \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

B4.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 13/2 \\ 0 \\ -2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -7 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

B5. (b) 14
(c) Idea: consider adding or removing suitable triangles. (or using cross product of vectors)

B6. (a) No: It is not the case that there is a pivot in each row
(b) for example $(0,0,0,0)$ and $(19,-18,7,1)$

B7. (a) Sometimes false. For example, $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, and $\boldsymbol{v}_3$ could all be zero.
(b) Always false. For example, all four vectors could be zero.
(c) Always true. The new vector cannot be zero.
(d) Always true. If $a_1\boldsymbol{v}_1+a_2\boldsymbol{v}_2+a_3\boldsymbol{v}_3+a_4\boldsymbol{v}_4=\mathbf{0}$, then $b_1\boldsymbol{v}_1+b_2(2\boldsymbol{v}_2)+b_3(3\boldsymbol{v}_3)+b_4(4\boldsymbol{v}_4)=\mathbf{0}$, where $b_i=a_i/i$. Further, b_i is nonzero whenever a_i is.