Exercises for Chapter 3

C1. Compute, where legal, $A+B$, AB, and BA for the following matrices:

$$A = \begin{bmatrix} 1 & -1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

C2. Compute, where legal, $3R^T$, R^2, and RS for the following matrices

$$R = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -2 \end{bmatrix} \qquad S = \begin{bmatrix} 0 & 1 \\ 2 & -3 \\ 1 & 1 \end{bmatrix}$$

C3. For two general square matrices A and B of the same size, simplify

$$\frac{1}{2}\left((2A^T + B)^T\right) - (-1)(-A^T - B^T)^T$$

C4. A 2×2 matrix A is called ***orange*** if $A^2 = I$. For example, the identity I itself is orange.

(a) Give a matrix, other than the identity, that is orange and has two zero entries.

(b) Give a matrix that is orange and none of its entries is zero.

C5. Matrix A is symmetric if $A = A^T$ and skew-symmetric if $A = -A^T$. Prove that every square matrix can be written as a sum of symmetric and a skew-symmetric matrix.

C6. Consider the matrix transform with matrix

$$A = \begin{bmatrix} 1 & -2 & 0 & 4 \\ 1 & 1 & -2 & 7 \\ 0 & 3 & -2 & 3 \end{bmatrix}$$

In each of the following, find a vector $\boldsymbol{x}$ whose image under the transform is $\boldsymbol{b}$, and determine if $\boldsymbol{x}$ is unique.

(i) $\boldsymbol{b} = (0, 0, 0)$ (ii) $\boldsymbol{b} = (3, 5, 2)$ (iii) $\boldsymbol{b} = (1, 1, 1)$

C7. In each of the following, draw vectors $\boldsymbol{u} = (1, 1)$ and $\boldsymbol{v} = (2, -1)$ and their image under the indicated transformation. Describe geometrically what the transformation does.

$$T_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \qquad T_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \qquad T_3 = \begin{bmatrix} 2 & 6 \\ 0 & 2 \end{bmatrix}$$

C8. Find the inverse of the following matrix:

$$E = \begin{bmatrix} 3 & 2 & 0 & 0 \\ 4 & 3 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 2 \end{bmatrix}$$

C9. Calculate the inverses of the following matrices, if they exist.

$$A = \begin{bmatrix} 4 & -1 \\ 3 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & 0 & 0 \\ -3 & 0 & 0 \\ -2 & 3 & 7 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 3 & 0 \\ 5 & 0 & 0 \end{bmatrix}$$

C10. Find the inverse of a general 2×2 matrix using row reduction.

C11. Determine the inverse of

$$W = \begin{bmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ y & z & 1 \end{bmatrix}$$

C12. Show that if A and B are $n \times n$ matrices and AB is invertible, then so is B. (Hint: give a formula for the inverse.)

Some Solutions

C1. $A + B$ is not possible.

$$AB = \begin{bmatrix} 3 \end{bmatrix} \qquad BA = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

C2. R^2 not possible.

$$3R^T = \begin{bmatrix} 3 & 3 \\ 3 & 0 \\ 3 & -6 \end{bmatrix} \qquad RS = \begin{bmatrix} 3 & -1 \\ -2 & -1 \end{bmatrix}$$

C3. $\frac{1}{2}B^T - B$.

C4. (a) For example $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} 0 & 2 \\ \frac{1}{2} & 0 \end{bmatrix}$

(b) For example $\begin{bmatrix} 1/2 & 1 \\ 3/4 & -1/2 \end{bmatrix}$

C5. Use the fact that $A + A^T$ is symmetric and $A - A^T$ is skew-symmetric and note that $A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$.

C6. (i) Many answers: e.g. $(0, 0, 0, 0)$
(ii) Many answer: e.g. $(13/3, 2/3, 0, 0)$
(iii) No solution

C7. T_1 maps $\boldsymbol{u}$ to $(0, 1)$ and $\boldsymbol{v}$ to $(0, -1)$. It projects onto y-axis.
T_2 maps $\boldsymbol{u}$ to $(0, \sqrt{2})$ and $\boldsymbol{v}$ to $(\frac{3}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. It rotates through 45 degrees.
T_3 maps $\boldsymbol{u}$ to $(8, 2)$ and $\boldsymbol{v}$ to $(-2, -2)$. It expands and shears.

C8.

$$E^{-1} = \begin{bmatrix} 3 & -2 & 0 & 0 \\ -4 & 3 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -3 & 2 \end{bmatrix}$$

C9.

$$A^{-1} = \frac{1}{23}\begin{bmatrix} 5 & 1 \\ -3 & 4 \end{bmatrix} \qquad B \text{ is not invertible} \qquad C^{-1} = \begin{bmatrix} 0 & 0 & \frac{1}{5} \\ 0 & \frac{1}{3} & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

C10. Let $\Delta = ad - bc$. Then assuming $a \neq 0$:

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & d - \frac{bc}{a} & \frac{-c}{a} & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|cc} a & 0 & 1 + \frac{bc}{\Delta} & \frac{-bc}{\Delta} \\ 0 & 1 & \frac{-c}{\Delta} & \frac{a}{\Delta} \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|cc} 1 & 0 & \frac{d}{\Delta} & \frac{-bc}{\Delta} \\ 0 & 1 & \frac{-c}{\Delta} & \frac{a}{\Delta} \end{array}\right]$$

C11.

$$W^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -x & 1 & 0 \\ -y + zx & -z & 1 \end{bmatrix}$$

C12. $B^{-1} = (AB)^{-1}A$.