Exercises for Chapter 4

D1. Calculate the following determinants:

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 0 & 4 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 7 & -4 & 22 & 1 \\ 1 & 2 & 33 & -1 \\ -1 & -2 & -33 & 1 \\ 23 & 0 & 4 & 15 \end{bmatrix} \qquad C = \begin{bmatrix} -1 & -1 & 3 \\ 3 & 4 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

D2. Assume that matrix G is 3×3 with determinant 7. Give the determinant of the following (or state that it is not uniquely determined):

(a) $5G^T$

(b) The block-matrix $\begin{bmatrix} G^2 & 0 \\ 0 & (G^3)^{-1} \end{bmatrix}$

(c) The matrix $\begin{bmatrix} g_{22} & g_{23} & g_{21} \\ g_{32} & g_{33} & g_{31} \\ g_{12} & g_{13} & g_{11} \end{bmatrix}$ where g_{ij} denotes the entry in row i column j of G.

D3. (a) Show that every permutation matrix has determinant 1 or -1 in three ways: (i) using the definition of determinant; (ii) using the cofactor expansion; and (iii) using the row reduction method.

(b) Show that the transpose of a permutation matrix is its inverse.

(c) Show that the product of two permutation matrices is a permutation matrix.

D4. Let M be a 100×100 matrix where every entry has absolute value 1. That is, every entry is either 1 or -1, but they need not all be the same. Prove that the determinant of M is even.

D5. Use a matrix transform to determine the area of the ellipse $3x^2 + 4y^2 = 5$. (Hint: it's a circular argument.)

D6. Using only the definition of a determinant we gave, show that if two rows of a matrix are identical then its determinant is 0.

Some Solutions

D1. $\det(A) = 24$ (Swap the 1st and 3rd rows; then swap the 3rd and 4th rows. The result is diagonal with $1, 2, 3, 4$ on diagonal)

$\det(B) = 0$ (Matrix is singular: 3rd row is negative of the 2nd)

$\det(C) = 21$

D2. (a) $5^3 \times 7$
(b) $\frac{1}{7}$
(c) 7 (use row and column interchanges)

D3. (a) (i) there is exactly one nonzero transversal; (ii) induction (iii) interchanges brings to the identity
(b) compute the product of matrix and transpose
(c) when multiplying any matrix A by permutation matrix P on the right, the result is a re-arrangement of columns of A; so if A is permutation, then so is AP.

D4. Do one step of row reduction. At this point, all entries except those in the first row will now be -2, 0, or 2. So all cofactors will be even.

D5. Need linear transformation A that maps unit circle to this ellipse. Matrix A takes $(1, 0)$ to $(\sqrt{5/3}, 0)$ and $(0, 1)$ to $(0, \sqrt{5/4})$; so those are the columns of A. The ellipse has area π times determinant of A, which is $5/\sqrt{12}$. Hence answer is $\frac{5\pi}{2\sqrt{3}}$.

D6. Assume first and second row are identical. Consider transversals using same two columns from first and second row. (Draw a picture.) Say one column has two a's and one column has two b's. Then transversals involving the values a and b come in pairs and have opposite signs, and therefore cancel. Repeat for all pairs of columns.