Exercises for Chapter 5

E1. Consider \( \mathbb{R}^3 \). Is each of the following a subspace? Explain.

(a) The set of all vectors with all entries positive.
(b) The \( x \)-axis.
(c) The set of all vectors such that the first entry is bigger than the second.
(d) The plane containing the points \((1,0,0), (0,1,0), \) and \((0,0,1)\).
(e) The set of all linear combinations of vectors \((1,0,0), (0,1,0), \) and \((0,0,1)\).

E2. For the vector space \( M_2 \) of all \( 2 \times 2 \) matrices, determine with justification whether each of the following subsets is a subspace.

(a) All matrices whose determinant is zero.
(b) The set of all linear combinations of matrices \(
\begin{bmatrix}
1 & 1 \\
2 & 3
\end{bmatrix}
\) and \(
\begin{bmatrix}
2 & 2 \\
4 & 6
\end{bmatrix}
\)
(c) All matrices whose entries are integers.

E3. Show that the axioms of a vector space imply that for any vector \( u \) it holds that \( 0u = 0 \).

E4. (a) Show that the intersection of two subspaces is always a subspace.
(b) Give an example that shows that the union of two subspaces is not necessarily a subspace.

E5. If \( A \) and \( B \) are matrices, show that the column space of the product \( AB \) is contained within the column space of \( A \).
Some Solutions

E1. (a) No (e.g. does not contain \(0\))
(b) Yes
(c) No (e.g. does not contain \(0\))
(d) No (e.g. does not contain \(0\))
(e) Yes

E2. (a) Not subspace. Not closed under addition: consider for example
\[
\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}
\]
(b) Subspace. Any span is a subspace.
(c) Not subspace. Not closed under scalar multiplication: consider for example \(\frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\)

E3. \(0u = (0 + 0)u = 0u + 0u\) (by Axiom 8). Subtract \(u\) from both sides (possible by Axiom 5) to get \(0 = 0u\).

E4. (a) Consider subspaces \(S\) and \(T\). Since zero vector in both, it’s in the intersection. Consider vectors \(u\) and \(v\) in the intersection. Since both \(S\) and \(T\) closed under addition, the sum \(u + v\) is in both and therefore in the intersection. Similarly closure under scalar multiplication.
(b) Let \(S\) be subspace of all multiples of \((1, 0)\) and \(T\) be subspace of all multiples of \((0, 1)\); then the union contains \((1, 0)\) and \((1, 0)\) but not their sum \((1, 1)\).

E5. By definition of matrix multiplication, each column of \(AB\) is a linear combination of the columns of \(A\). That is, each column of \(AB\) is in the column space of \(A\).