Exercises for Chapter 7

G1. For the vector space M_2 of all 2×2 matrices, determine with justification whether each of the following subsets is a subspace.

(a) All matrices whose determinant is zero.

(b) The set of all linear combinations of matrices $\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$ and $\begin{bmatrix} 2 & 2 \\ 4 & 6 \end{bmatrix}$

(c) All matrices whose entries are integers.

G2. Show that the axioms of a vector space imply that for any vector $\boldsymbol{u}$ it holds that $0\boldsymbol{u} = \mathbf{0}$.

G3. Show that the range of a linear transform $T : U \to V$ is a subspace of V.

G4. Describe all linear transforms from $\mathbb{R}$ to $\mathbb{R}$.

G5. Let L be the set of all linear transforms from $\mathbb{R}^3$ to $\mathbb{R}^2$.

(a) Verify that L is a vector space.

(b) Determine the dimension of L.

Some Solutions

G1. (a) Not subspace. Not closed under addition: consider for example $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

(b) Subspace. Any span is a subspace.

(c) Not subspace. Not closed under scalar multiplication: consider for example $\frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

G2. $0\boldsymbol{u} = (0+0)\boldsymbol{u} = 0\boldsymbol{u} + 0\boldsymbol{u}$ (by Axiom 8). Subtract $\boldsymbol{u}$ from both sides (possible by Axiom 5) to get $\mathbf{0} = 0\boldsymbol{u}$.

G3. We saw already that T maps $\mathbf{0}$ to $\mathbf{0}$; so the vector $\mathbf{0}$ is in the range. To show that the range is closed under addition, consider two vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ in the range. Then there are vectors $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ in U such that $T\boldsymbol{u}_1 = \boldsymbol{v}_1$ and $T\boldsymbol{u}_2 = \boldsymbol{v}_2$. Since U is closed under addition, we know that it contains $\boldsymbol{u}_1 + \boldsymbol{u}_2$. By the rules of a linear transform, $T(\boldsymbol{u}_1 + \boldsymbol{u}_2) = T(\boldsymbol{u}_1) + T(\boldsymbol{u}_2) = \boldsymbol{v}_1 + \boldsymbol{v}_2$. That is, the vector $\boldsymbol{v}_1 + \boldsymbol{v}_2$ is in the range. One can similarly show that the range is closed under scalar multiplication.

G4. Each linear transform has the form $T(x) = cx$ for all x in $\mathbb{R}$.

G5. (a) The zero transform acts as the zero of the vector space L. The sum of two transforms is another transform; the scalar multiple of a transform is another transform.

(b) L has dimension 6. The reasoning is that a transform is specifed if we know the values of $T(\boldsymbol{e}_1)$, $T(\boldsymbol{e}_2)$, and $T(\boldsymbol{e}_3)$. Each of these images is a vector in $\mathbb{R}^2$, and so has two degrees of freedom.