Exercises for Chapter 8

H1. For the matrix

$$E = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 3 & 1 \\ -1 & 3 & 0 \end{bmatrix}$$

(a) Give an eigenvector for the eigenvalue $\lambda = 1$.

(b) Determine the other two eigenvalues.

H2. Let T be the 100×10 matrix that has 0's on and below the diagonal, and 1's above the diagonal. Determine all eigenvalues and eigenvectors of T.

H3. Consider the following matrix.

$$C = \begin{bmatrix} 7 & 6 & 10 \\ 24 & 17 & 30 \\ -16 & -12 & -21 \end{bmatrix}$$

(a) Give a basis for the eigenspace for eigenvalue $\lambda = -1$ of C.

(b) Calculate the trace of matrix C.

(c) Give two distinct eigenvalues of C^{2019}.

H4. The Fibonacci matrix is defined as $F = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. Define a sequence of vectors by $\boldsymbol{f}_0 = (1, 1)$ and $\boldsymbol{f}_{i+1} = F\boldsymbol{f}$.

(a) Explain the relationship between the sequence of vectors and the Fibonacci sequence.

(b) Determine the eigenvalues of F.

(c) Use these ideas to approximate the 1000^{th} Fibonacci number.

H5. Find the eigenvalues and eigenvectors of the following matrix.

$$W = \begin{bmatrix} 7 & -9 \\ 4 & 7 \end{bmatrix}$$

H6. Show that if A is a 3×3 matrix with eigenvalues $0, 1, -1$, then it must be the case that $A^5 = A$.

Some Solutions

H1. (a) $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

(b) $0, 4$

H2. T has eigenvalue 0. Eigenvector $(1, 0, 0, \dots, 0)$.

H3. (a)

$$\{ \begin{bmatrix} -3/4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5/4 \\ 0 \\ 1 \end{bmatrix} \}$$

(b) 3

(c) Since $\lambda = -1$ has algebraic multiplicity 2, the third eigenvalue is $3 - 2(-1) = 5$. So the eigenvalues of C^{2019} are -1 and 5^{2019}.

H4. (a) $\boldsymbol{f}_1 = (2, 1)$; $\boldsymbol{f}_2 = (3, 2)$; $\boldsymbol{f}_3 = (5, 3)$.

(b) $\boldsymbol{f}_i$ has entries the $(i+1)^{\text{th}}$ and i^{th} Fibonacci numbers (assuming start with 1 as the 0^{th} Fibonacci number).

(c) Solve $(1 - \lambda)(-\lambda) - 1 = 0$. So $\lambda = (1 \pm \sqrt{5})/2$.

(d) Something like $(\frac{1+\sqrt{5}}{2})^{1000}$

H5. $7 + 6i$ with $(3i, 2)$; $7 - 6i$ with $(-3i, 2)$

H6. The matrix A has all distinct eigenvalues and so is similar to a diagonal matrix D with $0, 1, -1$ on the diagonal. It follows that $D^5 = D$.