Exercises for Chapter 9

I1. Compute $\boldsymbol{a} \cdot \boldsymbol{b}$, $\boldsymbol{a} \cdot \boldsymbol{c}$, and $\boldsymbol{b} \cdot \boldsymbol{c}$, when $\boldsymbol{a} = (1, -1, 1, -1)$, $\boldsymbol{b} = (-2, -2, 7, \pi)$, and $\boldsymbol{c} = (0, 0, 1, 0)$.

I2. Consider $\boldsymbol{v} = (4, 7)$.
(a) Give a unit vector in the same direction as $\boldsymbol{v}$.
(b) Give a unit vector orthogonal to $\boldsymbol{v}$.

I3. For each of the following triples, determine k such that it is orthogonal.
(a) $(1, 0, 0)$, $(0, 2, 1)$, $(0, 1, k)$
(b) $(1, 2, -2)$, $(2, 3, 4)$, $(0, 3, k)$
(c) $(k, k, 0, 0)$, $(k, 0, k, 0)$, $(0, 0, k, k)$

I4. Find a 4×4 matrix all of whose entries are ± 1 such that the columns are pairwise orthogonal.

I5. A matrix P is defined to be a ***projection matrix*** if $P^2 = P$ and $P = P^T$.

(a) Show that every eigenvalue of any projection matrix is 0 or 1.

(b) Show that if U is an invertible matrix, then $P_U = U(U^T U)^{-1} U^T$ is a projection matrix.

(c) Show that $\text{proj}_{\boldsymbol{y}}(U) = P_U \, \boldsymbol{y}$. That is, P_U is the matrix transform that projects $\boldsymbol{y}$ onto the columns of U.

I6. (a) Show that every orthonormal matrix U has determinant ± 1.
(b) Give an example U that has determinant 1 and contains no zeroes.
(c) Give an example U that has determinant -1 and contains no zeroes.

I7. Prove that if the columns of the square matrix F are orthonormal, then so are the rows. (Hint: consider F^{-1}.)

I8. Use Gram-Schmidt to produce an orthonormal basis for the space spanned by $(1, 1, 1, 0, 0)$, $(0, 1, 1, 1, 0)$ and $(0, 0, 1, 1, 1)$.

Some Solutions

I1. $\boldsymbol{a} \cdot \boldsymbol{b} = 7 - \pi$; $\boldsymbol{a} \cdot \boldsymbol{c} = 1$; $\boldsymbol{b} \cdot \boldsymbol{c} = 7$.

I2. (a) $(4/\sqrt{65}, 7/\sqrt{65})$
(a) $(7/\sqrt{65}, -4/\sqrt{65})$

I3. (a) $k = -2$
(b) DNE
(c) $k = 0$

I4. For example $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$

I5. (a) Assume $P\boldsymbol{v} = \lambda\boldsymbol{v}$ where $\boldsymbol{v} \neq \mathbf{0}$. Then $P^2\boldsymbol{v} = \lambda^2\boldsymbol{v}$. But we are given that $P^2 = P$. Therefore $\lambda^2 = \lambda$. This implies that λ is 0 or 1.
(b) $P_U^2 = \left(U(U^TU)^{-1}U^T\right)\left(U(U^TU)^{-1}U^T\right) = U(U^TU)^{-1}(U^TU)(U^TU)^{-1}U^T = P_U$. Similarly, $P_U^T = P_U$.
(c) ADD ME

I6. (a) Since $U^TU = I$, we have $\det U \cdot \det U = 1$.
(b) $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$
(b) $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

I7. Since columns orthonormal, we have $F^TF = I$. This means that $F^{-1} = F^T$. So we have $FF^T = I$ too. That is, the rows are orthonormal.

I8. $\frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\frac{1}{\sqrt{15}}\begin{bmatrix} -2 \\ 1 \\ 1 \\ 3 \\ 0 \end{bmatrix}$, and $\frac{1}{\sqrt{40}}\begin{bmatrix} 1 \\ -3 \\ 2 \\ 1 \\ 5 \end{bmatrix}$