

Wagering in Final Jeopardy

At the end of the television quiz program *Jeopardy!* comes *Final Jeopardy!*. The contestants are able to wager all or part of their score (or “dollars”) accumulated thus far on one final question. If they respond correctly (which in Jeopardy! reversed parlance means give the correct question) then their score is increased by the wager. If they are incorrect their score is decreased by that amount. We investigate here what their betting strategy should be.

On the program the topic for the final category is revealed and then the (usually) three contestants independently and secretly determine how much they wish to risk. After the bids are sealed, the final question is revealed. The players then write down their responses independently. These responses are then examined and the scores adjusted. The person with the highest score has his/her scores converted into dollars (something like ten thousand dollars on average), while the other players receive substantial consolation prizes. The winner returns the next day to play again.

In order to analyze the game we make some simplifying assumptions. We will assume that the sole objective of a player is to maximize his/her chance of winning and the actual amount is unimportant. This is probably reasonable. We will also assume that any real bet is allowed. On the actual program the contestants are restricted to multiples of 1 dollar but this does not change the character of the game much.

We are unable to solve the three person game. So we analyze only the case where there are two players. The three-player version seems to be considerably more complicated.

1 Discrete Strategies for Two players

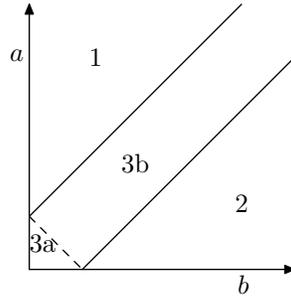
Since this is a zero-sum game, there are stable optimal strategies for the two players. In this section we show that there exist a pair of optimal strategies where each player chooses from a finite number of wagers.

Suppose that Player I has a score of 1 and Player II a score of $1 - \varepsilon$ for some

$\varepsilon > 0$. For definiteness we assume that if there is a tie then the first player wins. (This is a measure 0 event so inconsequential.)

Suppose Player I's wager is a and Player II's is b . Then there are three cases:

1. $a \geq b + \varepsilon$: Player I wins iff he responds correctly.
2. $b > a + \varepsilon$: Player I wins iff player II responds incorrectly.
- 3a. $a + b \leq \varepsilon$: Player I wins.
- 3b. $b - \varepsilon \leq a < b + \varepsilon$ and $a + b > \varepsilon$: Player I wins provided he answers correctly or his opponent answers incorrectly.



We see that the best situation for Player I is when his wager is about the same as his opponent's. Failing which he would prefer to outbid her if he has a better than 50-50 chance of getting the question right, and underbid her if she has a less than 50-50 chance of getting the question right.

Let us look first at the optimal strategy for Player I. Consider any wager of at least $1 - 2\varepsilon$. If he makes such a wager then he cannot be outbid. So only Cases 1 and 3 are possible. Since he prefers Case 3, Player I might as well minimize the amount of his wager. That is, any weight given to the range $[1 - 2\varepsilon, 1]$ might as well fall on $1 - 2\varepsilon$ itself. This reasoning is often employed on the actual program—the leader going into Final Jeopardy! frequently wagers just enough to ensure that she/he is the winner if responding correctly.

Consider Player II's optimal strategy and a wager of more than $1 - 3\varepsilon$. Then we are in either cases 2 or 3. If Player I bids less than $1 - 2\varepsilon$ then she can outbid him and avoid Case 3 by wagering $1 - \varepsilon$. If he does bid $1 - 2\varepsilon$ then Player II is

outfoxed and is in Case 3 no matter what. But Case 3b is preferable to 3a. So all the weight she places on the range $(1 - 3\varepsilon, 1 - \varepsilon]$ might as well be on $1 - \varepsilon$.

We can continue this process of discretization. Consider now Player I's wagers in the range $[1 - 4\varepsilon, 1 - 2\varepsilon)$. If Player II wagers $1 - \varepsilon$ then he is outbid. Otherwise we are in either cases 1 or 3. Since Player I prefers Case 3a to Case 3b, and the latter to Case 1, he should put all the weight from this interval on $1 - 4\varepsilon$.

By repeating this process enough times we can modify the players' strategies, while maintaining optimality, until we have a pair of strategies where Player I chooses his wager from $1 - 2\varepsilon, 1 - 4\varepsilon, \dots$ and 0, while Player II chooses her wager from $1 - \varepsilon, 1 - 3\varepsilon, \dots$

Let $l = \lfloor 1/\varepsilon \rfloor$ and $k = \lceil l/2 \rceil$. When l is even it follows that Player II's lowest potential wager is more than ε . So in this case any weight Player I might put on the wager 0 might as well be put on $1 - l\varepsilon$.

Thus we have shown that there exist optimal strategies for the two players in which each player has k options:

- If l is even then:
 Player I's choices are $\alpha_1 = 1 - 2\varepsilon, \alpha_2 = 1 - 4\varepsilon, \dots, \alpha_k = 1 - l\varepsilon$; and
 Player II's choices are $\beta_1 = 1 - \varepsilon, \beta_2 = 1 - 3\varepsilon, \dots, \beta_k = 1 - (l - 1)\varepsilon$.
- If l is odd then:
 Player I's choices are $\alpha_1 = 1 - 2\varepsilon, \alpha_2 = 1 - 4\varepsilon, \dots, \alpha_{k-1} = 1 - (l - 1)\varepsilon$ &
 $\alpha_k = 0$; and
 Player II's choices are $\beta_1 = 1 - \varepsilon, \beta_2 = 1 - 3\varepsilon, \dots, \beta_k = 1 - l\varepsilon$.

Suppose Player I bids α_i and Player II β_j . If $i > j$ then we are in Case 1 and if $i < j$ in Case 2. If $i = j$ then we are in Case 3b unless l is odd and $i = j = k$ when Case 3a is where we're at.

2 Solution for Two players

Suppose there is probability x that Player I responds correctly, probability z that Player II responds incorrectly, and probability y that either Player I responds correctly or Player II incorrectly (or both).

Consider first the case that l is odd. The payoff matrix for Player I, viewed as the row player, has y on the diagonal, x above the diagonal and z below the diagonal. This is a diagonally dominant square matrix with all entries positive. So we can solve the game given by the matrix by choosing a player's strategy to make his/her expected return independent of the opponent's strategy.

$$\begin{pmatrix} y & x & x & x \\ z & y & x & x \\ z & z & y & x \\ z & z & z & y \end{pmatrix}$$

Suppose Player I plays wager α_i with frequency p_i . Let $p = p_1$ and

$$\gamma = (y - x)/(y - z).$$

Then $p_i = \gamma^{i-1}p$. Since the frequencies sum to 1, it follows that $p = (\gamma - 1)/(\gamma^k - 1)$. The value of the game is thus:

$$z - \frac{x - z}{\gamma^k - 1}.$$

If l is even then the payoff matrix is similar except that the last diagonal entry is a 1. In this case $p_i = \gamma^{i-1}p$ for $1 \leq i \leq k - 1$, and $p_k = (y - x)\gamma^{k-2}p/(1 - z)$. It follows that $p = ((\gamma^{k-1} - 1)/(\gamma - 1) + (y - x)\gamma^{k-2}/(1 - z))^{-1}$ and the value of the game is

$$z + \frac{y - z}{(\gamma^{k-1} - 1)/(\gamma - 1) + (y - x)\gamma^{k-2}/(1 - z)}.$$

Example. Consider the case where $x = 0.6$, $y = 0.8$ and $z = 0.4$ and $\varepsilon = 0.29$. Then $l = 3$ and $k = 2$. One pair of optimal strategies is: Player I should wager high 3/4 of the time and nothing 1/4 of the time. Player II should wager all or nothing with 50–50 probability.