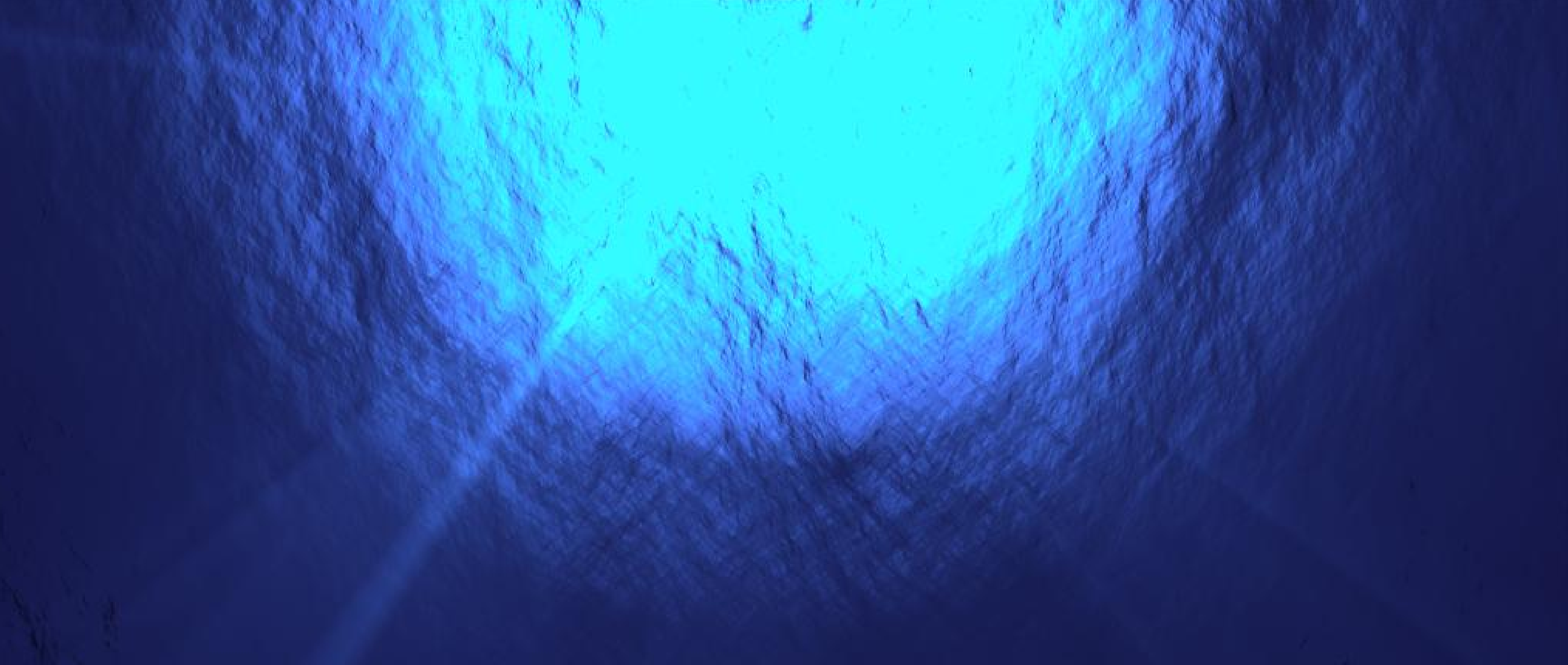

Simulating Ocean Surfaces

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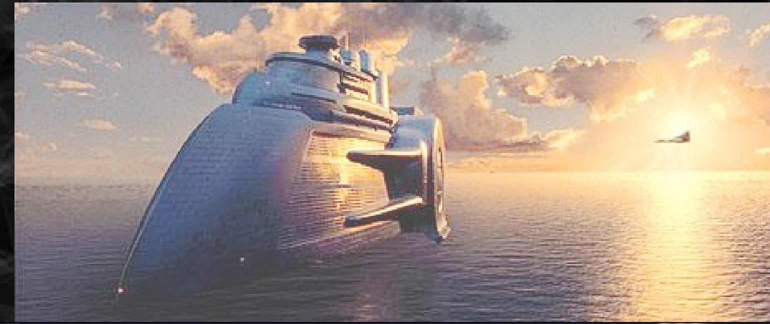
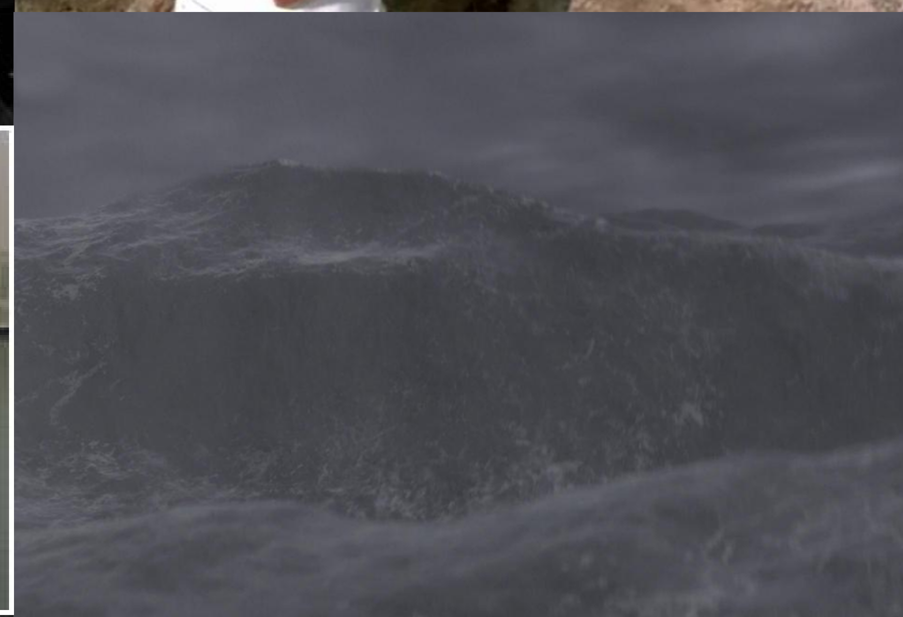
animation



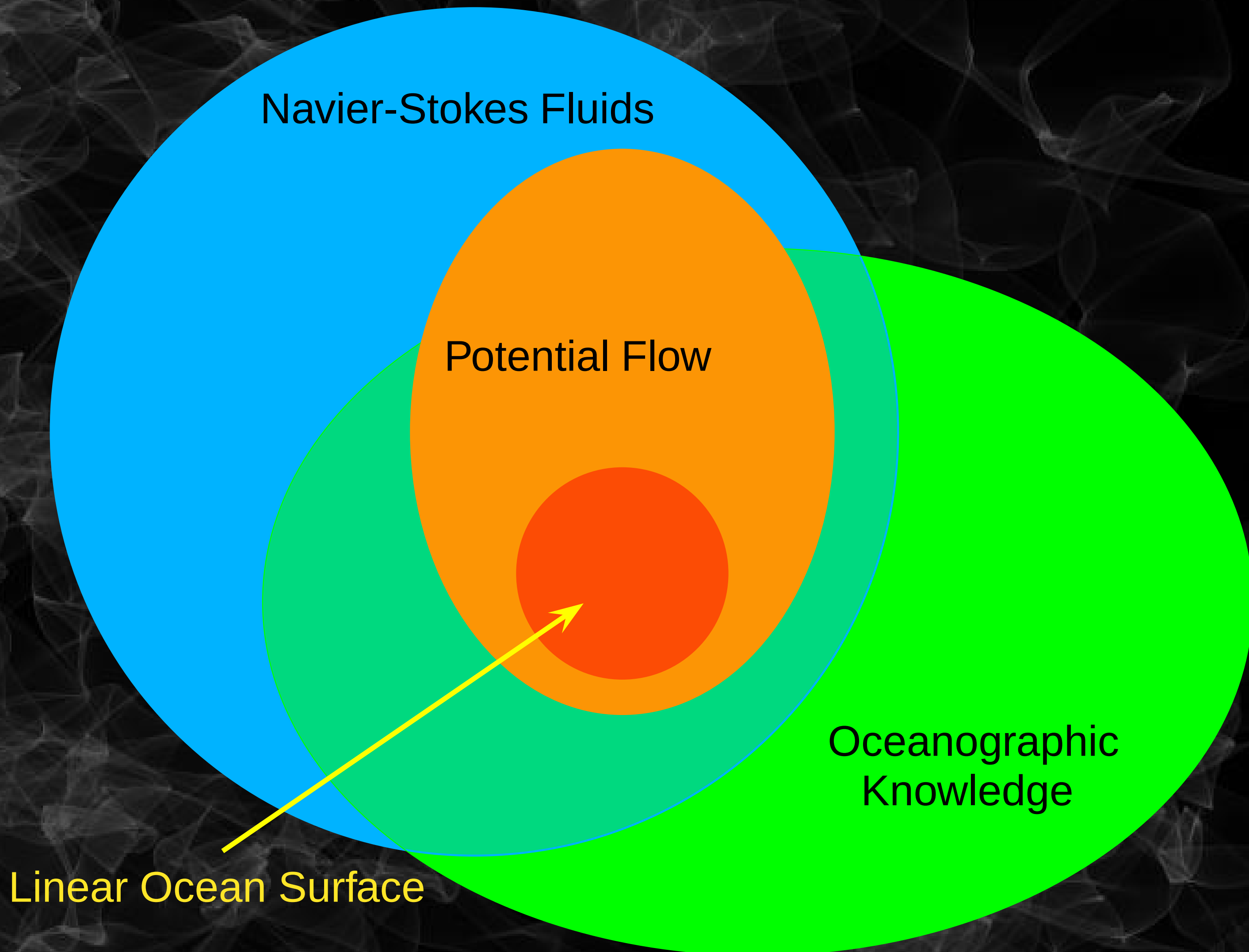
Waterworld
Truman Show
Hard Rain
Contact
Cast Away
Orange County
Myst III

13th Warrior
Titanic
Deep Blue Sea
Virus
World Is Not Enough
Pearl Harbor

Fifth Element
Double Jeopardy
Devil's Advocate
20k Leagues Under the Sea
13 Days
Moby Dick



First Principles to Phenomenology



Navier-Stokes Fluid Dynamics

Force Equation

$$\frac{\partial \mathbf{u}(\mathbf{x},t)}{\partial t} + \mathbf{u}(\mathbf{x},t)\cdot\nabla\mathbf{u}(\mathbf{x},t) + \nabla p(\mathbf{x},t)/\rho = -g\hat{\mathbf{y}} + \mathbf{F}$$

Mass Conservation

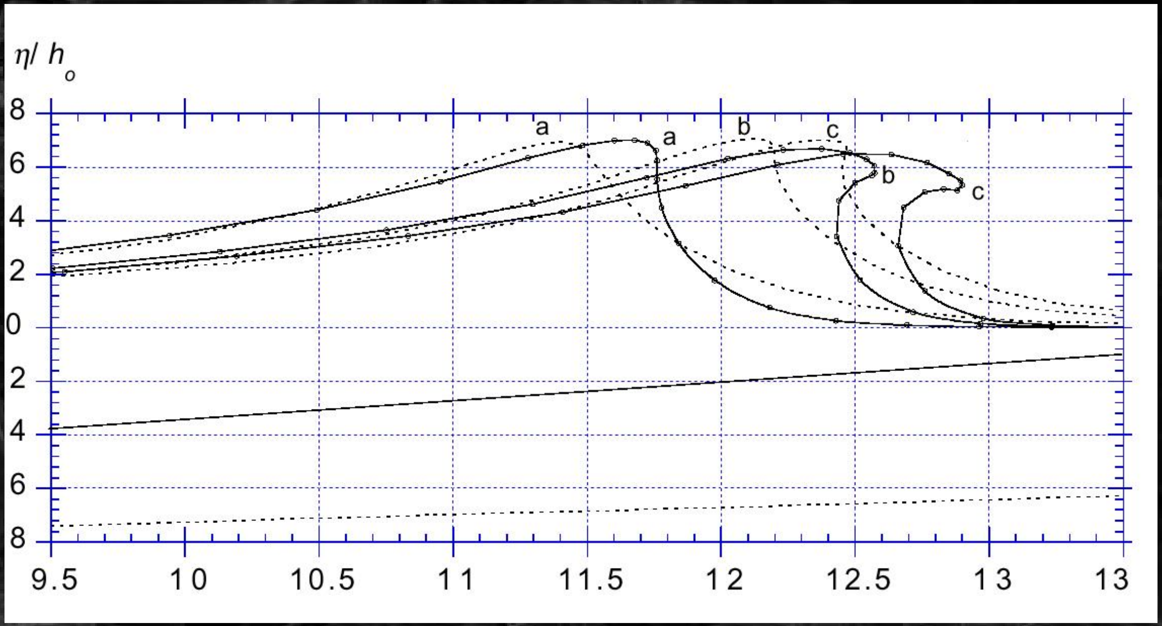
$$\nabla\cdot\mathbf{u}(\mathbf{x},t) = 0$$

Solve for functions of space and time:
- 3 velocity components
- pressure p
- density ρ distribution

Boundary conditions are important constraints

Very hard - Many scientitic careers built on this

Potential Flow



Grilli, Guyenne, Dias (2000)

Special Substitution $\mathbf{u} = \nabla\phi(\mathbf{x}, t)$

Transforms the Navier-Stokes equations into *Bernoulli's Equation*

$$\frac{\partial\phi(\mathbf{x},t)}{\partial t} + \frac{1}{2}\left|\nabla\phi(\mathbf{x},t)\right|^2 + \frac{p(\mathbf{x},t)}{\rho} + g\mathbf{x}\cdot\hat{\mathbf{y}} = 0$$

$$\nabla^2\phi(\mathbf{x},t) = 0$$

This problem is MUCH simpler computationally and mathematically.

Potential Flow

Special Substitution $\mathbf{u} = \nabla \phi(\mathbf{x}, t)$

Transforms the equations into

$$\frac{\partial \phi(\mathbf{x}, t)}{\partial t} + \frac{1}{2} |\nabla \phi(\mathbf{x}, t)|^2 + \frac{p(\mathbf{x}, t)}{\rho} + g\mathbf{x} \cdot \hat{\mathbf{y}} = 0$$

$$\nabla^2 \phi(\mathbf{x}, t) = 0$$

This problem is MUCH simpler computationally and mathematically.

Simplifying the Problem

Road to practicality - ocean surface:

- Simplify equations for relatively mild conditions
- Fill in gaps with oceanography.

Original dynamical equation at 3D points becomes linear equation on surface

$$\frac{\partial \phi(x,z,t)}{\partial t} = -gh(x,z,t)$$

Convert mass conservation to a vertical derivative computation

$$\hat{\mathbf{y}} \cdot \nabla \phi(x,z,t) \sim \left(\sqrt{-\nabla_H^2} \right) \phi = \left(\sqrt{-\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial z^2}} \right) \phi$$

Linearized Surface Waves

$$\frac{\partial h(x,z,t)}{\partial t} = \left(\sqrt{-\nabla_H^2}\right)\phi(x,z,t)$$

$$\frac{\partial \phi(x,z,t)}{\partial t} = -gh(x,z,t)$$

General solution easily computed in terms of Fourier Transforms

Solution for Linearized Surface Waves

General solution in terms of Fourier Transform

$$h(x,z,t) = \int_{-\infty}^{\infty} dk_x\, dk_z\, \tilde{h}(\mathbf{k},t)\, \exp\{i(k_x x + k_z z)\}$$

with the amplitude depending on the *dispersion relationship*

$$\omega_0(\mathbf{k}) = \sqrt{g\,|\mathbf{k}|}$$

$$\tilde{h}(\mathbf{k},t) = \tilde{h}_0(\mathbf{k}) \exp\{-i\omega_0(\mathbf{k})t\} + \tilde{h}_0^*(-\mathbf{k}) \exp\{i\omega_0(\mathbf{k})t\}$$

The complex amplitude $\tilde{h}_0(\mathbf{k})$ is arbitrary.

Examples of FFT surfaces

- ripple
- rain
- wake
- more complicated wake

Oceanography

- Think of the heights of the waves as a kind of random process
- Decades of detailed measurements support a statistical description of ocean waves.
- The wave height has a spectrum

$$\left\langle \left| \tilde{h}_0(\mathbf{k}) \right|^2 \right\rangle = P_0(\mathbf{k})$$

- Oceanographic models tie P_0 to environmental parameters like wind velocity, temperature, salinity, etc.

Models of Spectrum

- Wind speed V
- Wind direction vector $\hat{\mathbf{V}}$ (horizontal only)
- Wavelength of biggest waves $L = V^2/g$ (g=gravitational constant)
- Wavelength of smallest waves ℓ (user choice)

Parameterized Model Ocean Spectrum

$$P_0(\mathbf{k}) = \left|\hat{\mathbf{k}} \cdot \hat{\mathbf{V}}\right|^A \frac{\exp(-1/k^2L^2)}{k^B} \exp(-k^2\ell^2)$$

Typically, $A \approx 2$ and $B \approx 4$.

High Resolution Rendering

Sky reflection, upwelling light, sun glitter

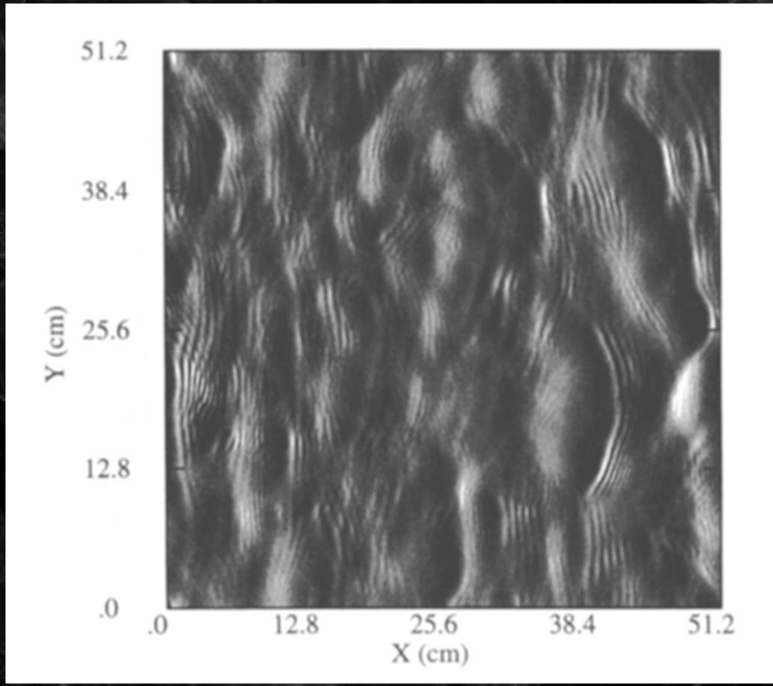
1 inch facets, 1 kilometer range

Hamiltonian Approach for Surface Waves

Miles, Milder, Henyey, ...

- If a crazy-looking surface operator like $\sqrt{-\nabla_H^2}$ is ok, the exact problem can be recast as a *canonical problem* with momentum ϕ and coordinate h in 2D.

- Milder has demonstrated numerically:
  - The onset of wave breaking
  - Accurate capillary wave interaction



- Henyey *et al.* introduced *Canonical Lie Transformations*:
  - Start with the solution of the linearized problem - (ϕ_0, h_0)
  - Introduce a continuous set of transformed fields - (ϕ_q, h_q)
  - The exact solution for surface waves is for $q = 1$.

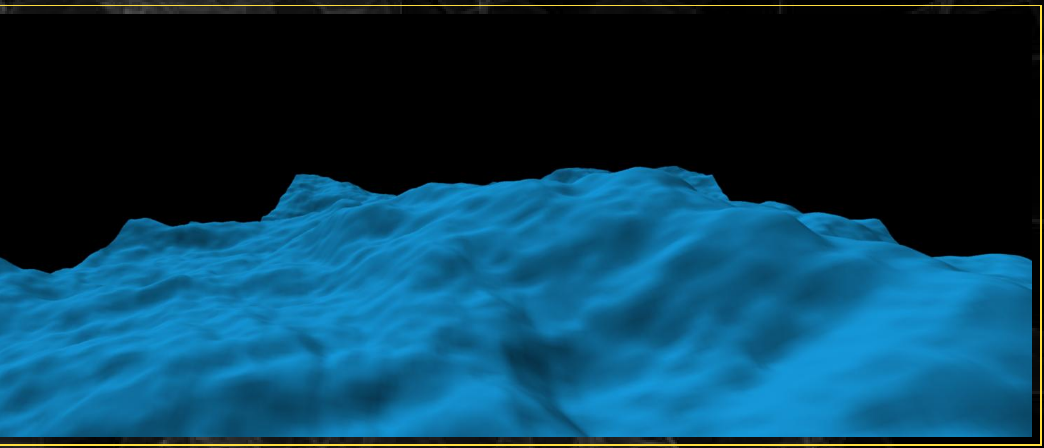
Choppy, Near-Breaking Waves

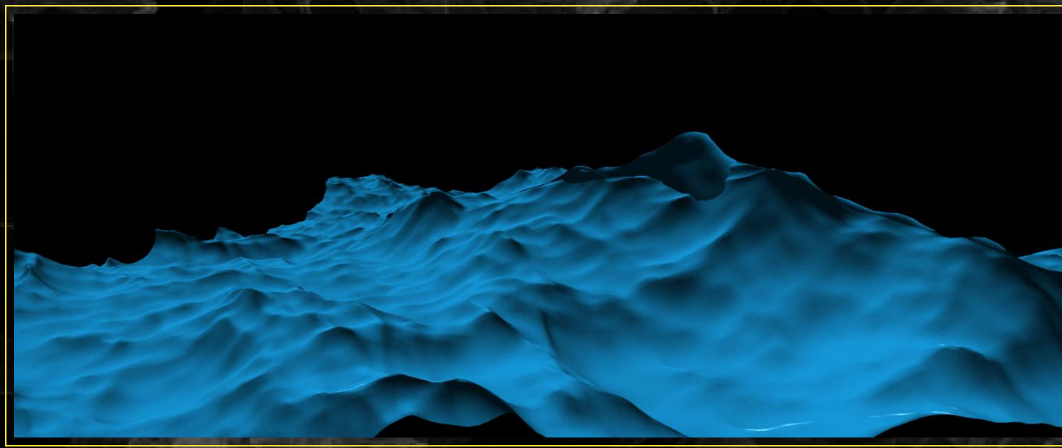
Horizontal velocity becomes important for distorting wave.

Wave at $\mathbf{x}$ morphs horizontally to the position $\mathbf{x} + \mathbf{D}(\mathbf{x}, t)$

$$\mathbf{D}(\mathbf{x}, t) = -\lambda \int d^2k \, \frac{i\mathbf{k}}{|\mathbf{k}|} \tilde{h}(\mathbf{k}, t) \; \exp\{i(k_x x + k_z z)\}$$

The factor λ allows artistic control over the magnitude of the morph.





Choppy Waves: Detecting Overlap

$$\mathbf{x} \to \mathbf{X}(\mathbf{x}, t) = \mathbf{x} + \mathbf{D}(\mathbf{x}, t)$$

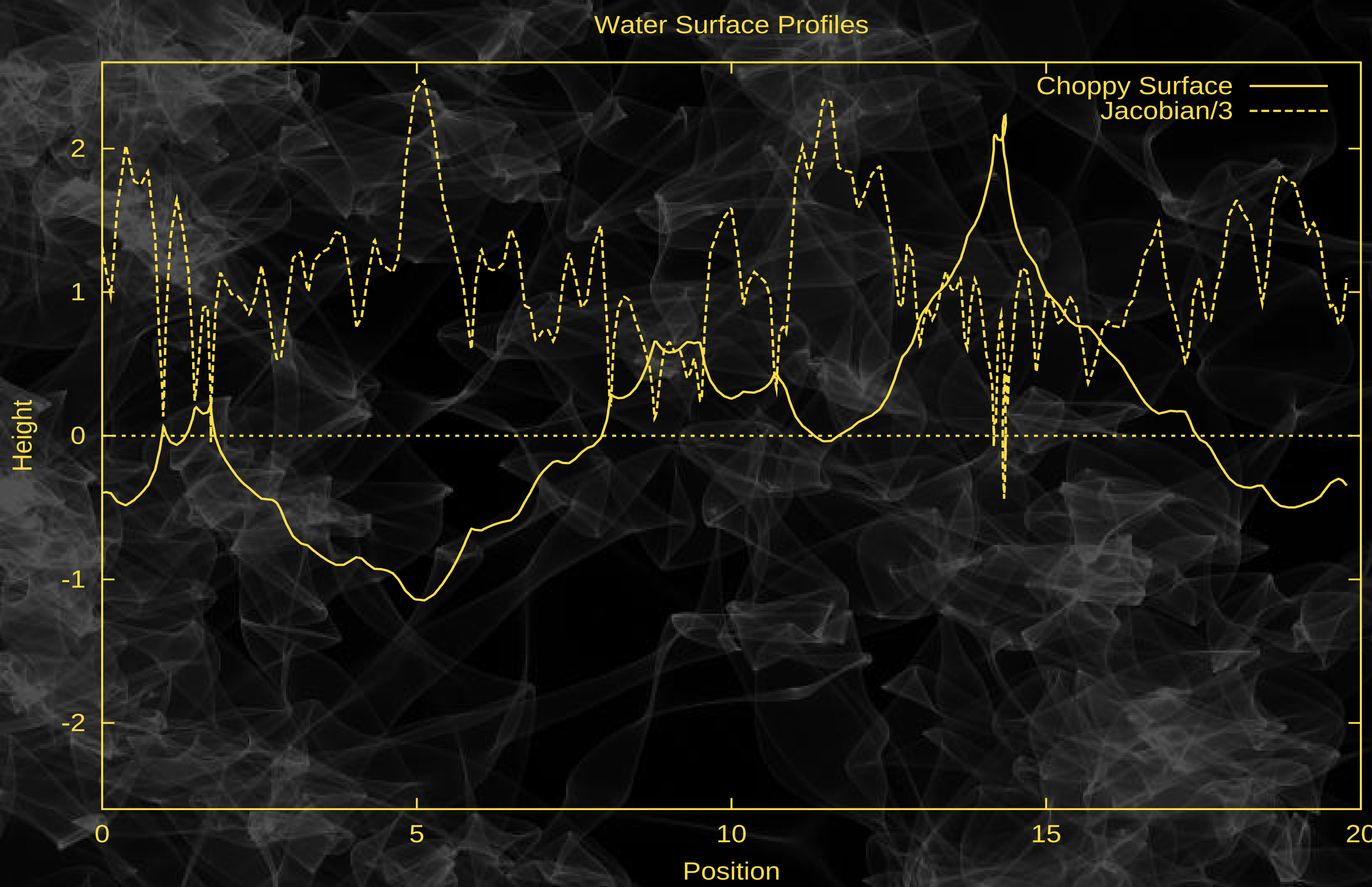
is unique and invertible as long as the surface does not intersect itself.

When the mapping intersects itself, it is not unique. The quantitative measure of this is the *Jacobian* matrix

$$J(\mathbf{x}, t) = \begin{bmatrix} \partial \mathbf{X}_x / \partial x & \partial \mathbf{X}_x / \partial z \\ \partial \mathbf{X}_z / \partial x & \partial \mathbf{X}_z / \partial z \end{bmatrix}$$

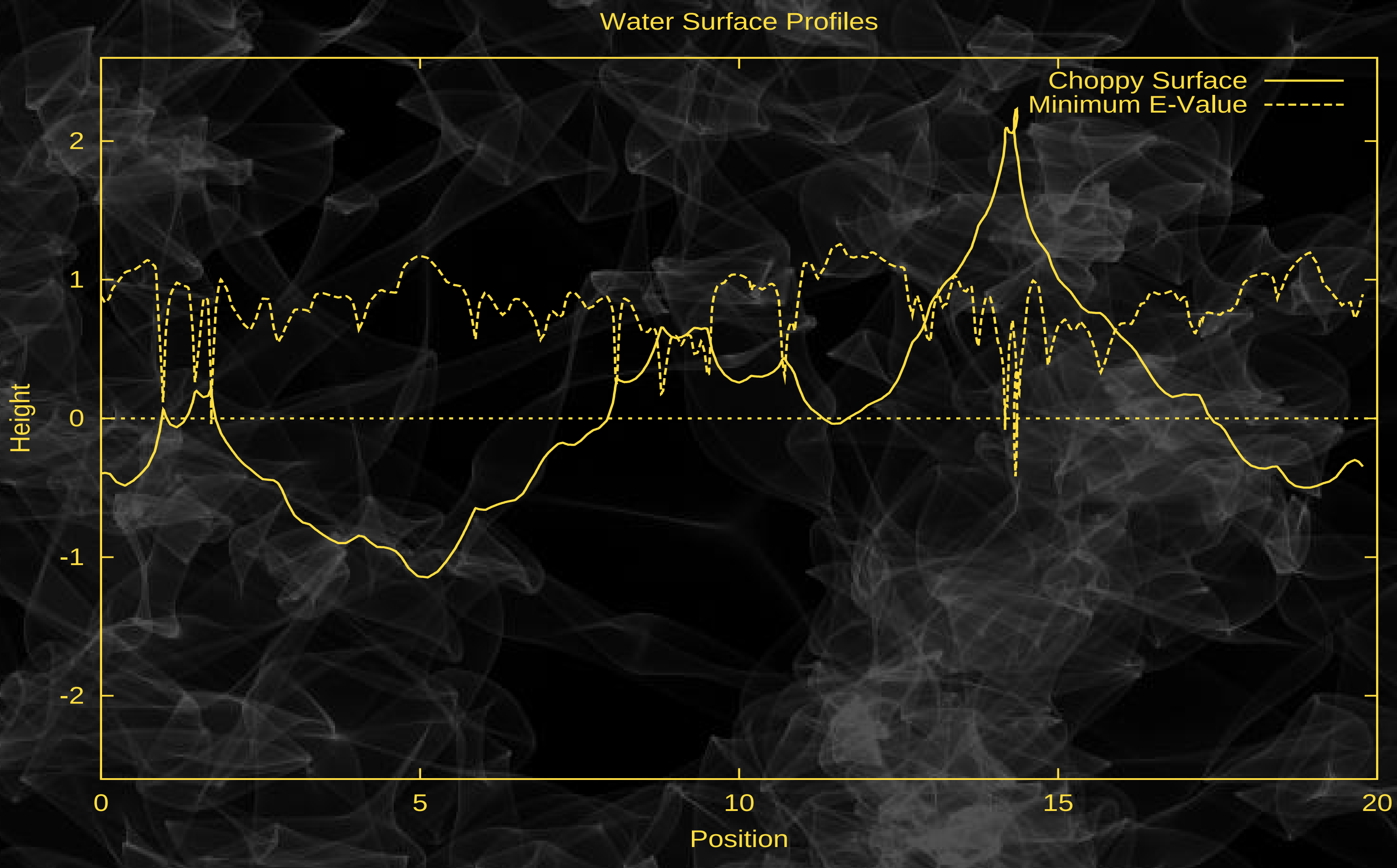
The signal that the surface intersects itself is

$$\det(J) \leq 0$$


Water Surface Profiles
Choppy Surface
Jacobian/3
Height
Position
2
1
0
-1
-2
0
5
10
15
20

Directional Character

- The 2×2 Jacobian matrix describes folding in two directions.
- Eigenvalues and eigenvectors tell the amounts of folding and the fold directions.
- Minimum eigenvalue is the most folding, and minimum eigenvector is the direction of folding.



Simple Spray Algorithm

- Pick a point on the surface at random
- Emit a spray particle if $J_- < J_T$ threshold
- Particle initial direction ($\hat{\mathbf{n}}$ = surface normal)

$$\hat{\mathbf{v}} = \frac{(J_T - J_-)\hat{\mathbf{e}}_- + \hat{\mathbf{n}}}{\sqrt{1 + (J_T - J_-)^2}} \qquad (1)$$

- Particle initial speed from a half-gaussian distribution with mean proportional to $J_T - J_-$.
- Simple particle dynamics: gravity and wind drag

animation

Summary

- FFT-based random ocean surfaces are fast to build, realistic, and flexible.
- Based on a mixture of theory and experimental phenomenology.
- Used alot in professional productions.
- Real-time capable for games
- Lots of room for more complex behaviors.
- Future: dynamic interactions

Latest/expanded version of course notes and slides:

http://home1.gte.net/tssndrf/index.html