Quick Refresher on Algebra, Probability,…

Progressions (AP, GP, HP)

Arithmetic Progression (AP): A sequence of integers is called an arithmetic progression (AP) if the difference between any two consecutive terms is always same (constant). It means that next number in the series is computed by adding a fixed number to the previous number in the series. This fixed number is called the common difference. For example, 2,4,6,8,10 is an AP because difference between any two consecutive terms in the series (common difference) is same (4 – 2 = 6 – 4 = 8 – 6 = 10 – 8 = 2).

If 'a' is the first term and 'd' is the common difference,

- n^{th} term of an AP = a + (n-1) d
- **Arithmetic Mean** = Sum of all terms in the AP / Number of terms in the AP
- Sum of 'n' terms of an AP = 0.5 n (first term + last term) = 0.5 n [2a + (n-1) d]

Geometric Progression (GP) : A sequence of numbers is called a geometric progression if the ratio of any two consecutive terms is always same. In simple terms, it means that next number in the series is calculated by multiplying a fixed number to the previous number in the series. This fixed number is called the common ratio. For example, 2,4,8,16 is a GP because ratio of any two consecutive terms in the series (common difference) is same (4 / 2 = 8 / 4 = 16 / 8 = 2).

If 'a' is the first term and 'r' is the common ratio,

- n^{th} term of a GP = a r^{n-1}
- **Geometric Mean** = nth root of product of n terms in the GP
- Sum of 'n' terms of a GP (r < 1) = [a (1 – r^n)] / [1 – r]
- Sum of 'n' terms of a GP (r = 1) = na
- Sum of 'n' terms of a GP (r > 1) = [a (r^n – 1)] / [r – 1]
- Sum of infinite terms of a GP (r < 1) = (a) / (1 – r)

Note: Computing the sum is easy too. Say the sum is S_n. Then, do the following to get S_n:

$S_n = a + ar + ar^2 + \ldots + ar^{n-1}$

$rS_n = \quad ar + ar^2 + \ldots + ar^{n-1} + ar^n$

Subtract the second one from the first we get $(1 - r)\, S_n = a(1 - r^n) \rightarrow$ we get S_n except when r = 1; but when r = 1, all terms are a and so $S_n = na$.

Harmonic Progression (HP): A sequence *an* of real numbers is a **harmonic progression** (HP) if any term in the sequence is the harmonic mean of its two neighbors.

The **harmonic mean** is calculated as the number N of values divided by the sum of the reciprocal of the values (1 over each value).

- Harmonic Mean = $N / (1/x_1 + 1/x_2 + \ldots + 1/x_N)$

If there are just two values (x_1 and x_2), a simplified calculation of the harmonic mean can be calculated as: Harmonic Mean = $(2 * x_1 * x_2) / (x_1 + x_2)$

The harmonic mean is the appropriate mean if the data is comprised of rates.

Recall that a rate is the ratio between two quantities with different measures, e.g., speed, acceleration, frequency, etc.

In machine learning, we have rates when evaluating models, such as the true positive rate or the false positive rate in predictions. The harmonic mean does not take rates with a negative or zero value, e.g., all rates must be positive.

One common example of the use of the harmonic mean in machine learning is in the calculation of the F-Measure (also the F1-Measure or the Fbeta-Measure); that is a model evaluation metric that is calculated as the harmonic mean of the precision and recall metrics.

A sequence a_n of real numbers is a **harmonic progression** (HP) if any term in the sequence is the harmonic mean of its two neighbors.

Theorem: A sequence is a harmonic progression if and only if its terms are the reciprocals of an arithmetic progression that doesn't contain 0.

Proof: The harmonic mean of $1/\{a+(n-1)d\}$ and $1/\{a+(n+d)\}$ is

$2/\{a+(n-1)d + a+(n=1)d\} = 2/(2a+2d) = 1/(a + nd)$ which proves the "if" part of the statement. The "only if" part is similar and left as an exercise: write the first two terms as 1/a and 1/(a+d), and show that the third term is 1/(a+2d), the fourth term is 1/(a+3d), and so on by induction.

Logarithms:

> Logarithms and Exponents**:** $\mathbf{\log_b a = c \text{ if } a = b^c}$. As is the custom in the computing literature, we omit writing the base b of the logarithm when b = 2. For example, log 1024 = 10 (or, lg 1024 =10, binary logarithm). We need to remember the following simple rules (remember how to prove them):
>
> $\log_b ac = \log_b a + \log_b c; \quad \log_b a/c = \log_b a - \log_b c; \log^k (n) = (\log n)^k$

$\log_b a^c = c\log_b a$; $\log_b a = (\log_c a)/(\log_c b)$;

$(b^a)^c = b^{ac}$; $b^a b^c = b^{a+c}$; $b^a / b^c = b^{a-c}$;

$\log_b(1/a) = -\log_b a$; $\log_b a = 1/\log_a b$;

$$b^{\log_c a} = a^{\log_c b}$$

Modulo Numbers:

- Given two positive integers, m and n, we define m modulo n (or, m mod n) as the remainder of the integer division of m by n [n is the modulus], i.e., 8 mod 3 = 2, 16 mod 4 = 0.
- **Modulo arithmetic**: Explain the basics, like (8 + 9) mod 7 = 3 = $8_7 + 9_7$, modulo n numbers are (0, 1, 2, …, n – 1).
- **Integers Z = {…, -2, -1, 0, 1, 2,}**
- Consider the integers Z. For a, b ∈ Z, we say that b divides a, or that a is divisible by b, if there exists c ∈ Z such that a = bc. If b divides a, then b is called a divisor of a, and we write a|b.
- Fix a +ve integer N (called **modulus**): for two integers a and b we write a = b (mod N) if N divides b − a, and we say that a and b are **congruent modulo N**.
- Consider (mod N) as a postfix operator on an integer which returns the smallest non-negative value equal to the argument modulo N, e.g., 16 (mod 5) = 1, -16 (mod 5) = 4; note the difference with % operator in C or Java; (-3) mod 2 = 1 while (-3)%2 = -1.
- Define the set N ={0,1,2, ..,N-1} = set of all possible remainders mod N. Addition and multiplication are defined the usual way on N. (13+12) mod 15 = 10, (5*6) mod 15 = 5, (8+4) mod 15 = 12, etc.
- Addition and multiplication mod N work almost the same as arithmetic over the reals or the integers. (1) Addition is closed, commutative, associative with identity 0; (2) multiplication is closed, commutative, associative with identity 1; (3) additive and multiplicative identities always exist; (4) multiplication and addition satisfy the distributive law.

Solve Quadratic Equations:

Consider the equation ax2 + bx + c = 0 $\Rightarrow 4a^2x^2 + 4abx + 4ac = 0 \Rightarrow (2ax)^2 + 2ax.b + b^2 = b^2 - 4ac \Rightarrow (2ax + b)^2 = b^2 - 4ac \Rightarrow 2ax + b = \sqrt{(b^2 - 4ac)} \Rightarrow x = [-b \pm [\sqrt{(b^2 - 4ac)}]/2a$

EXAMPLE

If the 3^{rd} term of an HP is 3 and the 10^{th} term of the same HP is 10, what is the maximum possible length of the HP?

The reciprocals of the HP form an arithmetic progression $a, a+d, a+2d, \ldots$. Then $a+2d=\frac{1}{3}$ and $a+9d=\frac{1}{10}$.

This gives $a=\frac{2}{5}$, $d=-\frac{1}{30}$. Note that $a+12d=0$, so the arithmetic progression can only have 12 terms before it contains 0. So the answer is 12. □

(Exercise: If the m^{th} term is m and the n^{th} term is n, the answer is $m+n-1$.)

EXAMPLE

If the first 2 terms of a harmonic progression a_n are $\frac{1}{19}$ and $\frac{1}{17}$, find the maximum partial sum $\sum_{n=1}^{k} a_n$.

The terms of the HP are

$$\frac{1}{19}, \frac{1}{17}, \frac{1}{15}, \cdots, \frac{1}{3}, \frac{1}{1}, \frac{1}{-1}, \frac{1}{-3}, \cdots$$

So the maximum partial sum is

$$\frac{1}{19}+\frac{1}{17}+\frac{1}{15}+\cdots+\frac{1}{1} \approx 2.13. \square$$

EXAMPLE

If the sum of the first 2 terms of an HP is $\frac{17}{70}$, the sum of the next 2 terms is $\frac{5}{4}$, and the sum of the following 2 terms is $\frac{-7}{10}$, find the sum of the following 2 terms (again).

According to the problem statement,

$$\frac{1}{a}+\frac{1}{(a+d)}=\frac{17}{70} \qquad (1)$$

$$\frac{1}{(a+2d)}+\frac{1}{(a+3d)}=\frac{5}{4} \qquad (2)$$

$$\frac{1}{(a+4d)}+\frac{1}{(a+5d)}=\frac{-7}{10}. \qquad (3)$$

Solving any two of (1), (2), and (3) simultaneously gives $a = 10$ and $d = -3$.

Therefore, the answer is

$$\frac{1}{(a+6d)} + \frac{1}{(a+7d)} = \frac{-19}{88}. \square$$

TRY IT YOURSELF

Given that

- a, b, c form an arithmetic progression,
- b, c, d form a geometric progression, and
- c, d, e form a harmonic progression,

what progression will a, c, e form?

- ○ Harmonic progression
- ○ Arithmetic Progression
- ○ None of these choices
- ○ Geometric Progression

Solve Quadratic Equations:

Consider the equation ax2 + bx + c = 0 $\Rightarrow$ $4a^2x^2 + 4abx + 4ac = 0 \Rightarrow (2ax)^2 + 2ax.b + b^2 = b^2 - 4ac \Rightarrow (2ax + b)^2 = b^2 - 4ac \Rightarrow 2ax + b = \sqrt{(b^2 - 4ac)} \Rightarrow x = [-b \pm [\sqrt{(b^2 - 4ac)}]/2a$