WHAT DOES EUCLID HAVE TO SAY ABOUT THE FOUNDATIONS OF COMPUTER SCIENCE?

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ABSTRACT. It is fair to say that Euclid's *Elements* has been a driving factor in the development of mathematics and mathematical logic for twenty-three centuries. The author's own love affair with mathematics and logic start with the *Elements*.

But who was the man? Do we know anything about his life? What about his times? The contemporary view of Euclid is much different than the man presented in older histories. And what would Euclid say about the status of computer science with its rules about everything?

1. Motivations

Many older mathematicians may have started their love affair with mathematics with a high school course in plane geometry. Until recently, plane geometry was taught from translations of the text such as by Heath [5]. In my own case, it was the neat, clean picture presented with its neat, clean proofs. I would guess that over the twenty-three hundred year history of the text, Euclid must be the all-time, best selling mathematics textbook author. At the turn of the last century, David Hilbert also found a love of Euclid. So much so that Hilbert was moved to propose his Formalist program for mathematics. Grossly over-simplified, Hilbert proposed that mathematics was a "meaningless game" in that the rules of logic preordained the results independent of any interpretation.

Hilbert was rather vehemently opposed by L. E. J. Brouwer. Brouwer proposed that it was the intuition of the mathematician who used a personal language to develop mathematics; this language always has meaning [14]. A focus of Brouwer was the concept of existence. Pretending to

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use an object that has not been constructed is called *impredication*. Henrí Poincaré said impredication was the root of all mathematical evil. Hilbert's program eventually would fail, done in by Gödel's Incompleteness Theorem. But by no means did this establish intuitionism as the wave of the future.

Intuitionism has several facets; it is the existence (no impredication) facet that brought me to study absolute geometry (part of Book I) with an eye to the history and philosophy of mathematics in Euclidean times. The cause for this was an article by von Plato [15] who proposed constructive axioms for geometry[1]Briefly, the difference between constructive and platonic can be stated in terms of Plato and his student Aristotle. This story is well-known in philosophy. Hopefully, not oversimplifying, the question is over existence.

Platonic mathematics postulates that things just exist. For example, the uncountable set of real numbers exists as an ideal and we can treat them as if we had access to this set. Essentially, this is a statement of Plato's philosophy. Plato's most famous pupil, Aristotle, disagreed, saying that things must be shown to exist by some method or other. Aristotle's stand is often called "empiricism." In modern garb, the argument was between Hilbert's formalism and Brouwer's intuitionism. It is said that "All mathematicians are intuitionists on Sunday and Platonist the rest of the week." Errett Bishop is the most recent proponent [1]. I find the intuitionistic argument far more satisfying in computer science: you cannot compute with something until you have created it.

According to the the Platonist, Euclid was a platonic mathematician — go figure. This view was popularized primarily by Proclus who once headed Plato's Academy and wrote an extensive commentary on the *Elements*[5]. Unfortunately, Proclus lived about six hundred years after Euclid. The history of the platonic *versus* constructionist views are well documented in [2, 6] and the details

[1]I use the terms *intuitionism* and *constructivism* interchangeably. This may not be technically correct but the term *construction* is an inherent part of the geometric discussion

Alexander of Aphrodisias	Amphinomus	Anaxgoras
Antiphon	Aristotle	Bryson
Democitus	Diocles	Eudemus
Eutocius of Aokolon	Hero	Nicomedes
Pappus	Plutarch	Plato
Proclus	Protagoras	Simplicius
	Theon of Smyrna	

FIGURE 1. Major Commentators on Geometry

are outside our story other than to say conventional wisdom would have said that the constructionist view was not Euclid's.

The received story of Euclid is a jumble of comments collected in many contexts. To hear the philosophers tell it, Euclid was as much a philosopher as mathematician. Many logicians talked of a grand scheme by Euclid to set the foundations of mathematics and mathematical logic. But was this his goal? And what was the intellectual climate preceding and during Euclid's time (even though we do not know precisely when that was)? There is confusion over what commentators have read into the text and what the ancient geometers might themselves have believed. Our problem was to understand the "true" Euclid.

Section 2 looks at the men and their times. Section 3 looks at the text of the *Elements* itself. What might be Euclid's view of how geometry is conducted in Section 4.

2. THE ANCIENTS

There seem to be far more ancient commentators (Fig. 1) on geometry than there are investigators (Fig 2). The purpose of this section is to identify those people who probably had a direct influence on the technical content of the *Elements*.

Mid 5th Century to Mid 4th Century

Hippocrates of Chios   Oenipides of Chios   Eratosthenes of Cyrene

Time of Plato and Aristotle

Archytas of Tarentum   Dinostratus   Eudoxus of Cnidus

Leodamus   Leon   Menaechmus

Theaetetus of Athens

Post Euclid

Appolonius   Archimedes

FIGURE 2. Major Contributors to Geometry

2.1. **The Men Themselves.** Of all those mentioned in the history of geometry there are four pre-Euclideans whom I will mention here. There are two groups: one in the 4th Century and the other in the time of Plato in the 3rd Century.

2.1.1. *4th Century Investigators.* In the 4th century BC, Hippocrates and Oenopides figure prominently in the century preceding the establishment of Plato's Academy. Oenopides is reputed to be the person who required only rulers and compasses be used. In several places, Knorr [9] expresses doubt that such a requirement was ever made. It seems highly unlikely that the early geometers put any artificial impediments in the way of development [9, p. 345]. Hippocrates was, apparently, *the* geometer of the 4th century. He added significantly to the knowledge of his time, but he was also considered the *first compiler* of geometry [7, p. 179].

2.1.2. *Plato's Academy.* After Hippocrates and Oenopides there is a lull in geometry research but there is also the founding of Plato's Academy. Plato himself was not a mathematician and the Academy was not a hotbed of mathematical research. Plato is known to have commented that the methods of geometry were not abstract enough; several centuries later, Carpus would complain that the methods were too abstract to be useful [9, p. 364–365]. The Academy did make a significant turn towards mathematics when Eudoxus joined it.

Eudoxus. Eudoxus is most remembered for his work on approximations to limits, or the so-called "method of exhaustion." He was also a consummate geometer. How much did he know about constructive methods? There is a group of historians that place heavy emphasis on religious motives for solving geometric problems. Seidenberg [11, 12] calls these "peg and cord" methods. These techniques were known throughout Western Asia. The story is that the priests of various religions solved several important problems constructively and the Greek mathematicians took it from there. Interestingly enough, Eudoxus *did* spend time in Egypt studying astronomy with the priests. The apparent impredication in Theorem 1, Book 1 can be seen differently through the methods discussed in Seidenberg [11, 12]. One need not invoke a continuity principle at all, but simply observe the use of cord and peg methods. On the other hand, religion was probably just another application [9, p. 16].

Eudoxus [6] is given credit for developing deductive reasoning. He also seems to have made a distinction between magnitudes and numbers. While numbers were prior and ruled by "common notions", magnitudes were not. Magnitudes were ruled by ratios and proportions.

Menaechmus. Menaechmus was at the Academy at the same time as Eudoxus, probably as a student, then as a teacher. Menaechmus puts the priority to problems over theorems [9, p. 76, p. 351]. There are two types of problems [9, p. 359]:

A. Those that lead to the determination by certain figures that result in another figure and
B. Those that lead to the construction of a figure with specified properties.

Name	Dates
Socrates	469 BC to 399 BC
Plato	c. 428 BC to c. 347 BC
Eudoxus	408 BC to 355 BC
Aristotle	384 BC to 322 BC
Euclid	No firm guesses: 325 BC to 300 BC
Archimedes	287 BC to 212 BC
Theon	c. 335 AD to 395 AD
Proclus	410 AD to 485 AD

FIGURE 3. Chronology of Important Greeks

Menaechmus seems to be the first recorded intuitionist focusing on establishing existence.

Euclid. Much controversy could be avoided if we knew more about Euclid the man. But Euclid is an unknown figure. Toomer [13] says "the biographical data linking Euclid with Alexandria and Ptolemy I are worthless references by late authors (Pappus and Proclus) who seem to have no more information than we do." Knorr is kinder [9, p. 138]. "The biographical data on Euclid is meager." Be that as it may, the dates (Fig. 3) for the main characters put Euclid in some interesting company.

As for Euclid himself, it is better to think of him [9, p 138] as a "compiler and an effective teacher." His contributions were seminal to the development of problem solving in the 3rd Century BC [9, p 138]. All references seem to agree that Euclid was not himself a terrific geometer. Despite this, one must wryly note that he did write the all time best seller in geometry if not all of mathematics. And this is a teaching text and not a research tract to boot!

The lack of information about Euclid leads to many (apparently invented) stories about his philosophical leanings. I was taught he was a Platonist. Given the chronology in Fig. 3 one might be inclined to accept that if one has some idea of Euclid's age when he did this work.

But, he also might well have known Aristotle and been more oriented towards his school. The habit of later commentators putting their own spin in earlier events is all too common. We have all heard the tale of Hippasus doomed to be shipwrecked for having divulged the secret of the irrational to the uninitiated. "We must be quite skeptical of placing much stock in interpretations on unsubstantiated stories. [9, p. 88]" We return to this in Section 4.

3. Euclid's *Elements*

Ultimately, our understanding rests on the *Elements*. There is the document itself and there are the organization and methods of Euclid.

3.1. **The Text.**

The best precis of the *Elements* itself is,

> "Yet for all its sophistication of its logical structure and intricacy of some of its constructions, . . . , the *Elements* is predominantly a treatise of an introductory sort, as its name implies. The researches indicated were initiated decades before Euclid by Hippocrates, Thaetetus and Eudoxus and advanced by their successors." [9, p. 102]

For much of history, the recension of Theon [13] was the basis of knowledge about the *Elements*. In 1814, a copy of the *Elements* was found in the Vatican library. What this means is that until 1814, all discussion about Euclid was really about Theon's annotated version. Clearly, *Elements* is not an exhaustive inventory of methods, problems, or results available to Euclid [9, p. 92].

3.2. **The Organization of the *Elements*.**

Because the text is essentially a textbook, Euclid's formalized organization probably did not follow the order of discovery [9, p. 71]. Given that the results are not Euclid's personal research and that he was writing a teaching document, we can assume that he chose those proofs for their pedagogical importance. This view is enhanced when you carefully consider Book I: Theorem 22 starts afresh from first principles.

A Note on Notes. The convention to talking about theorems in Euclid is to quote the book first and theorem number second. For example, *I.22* is theorem 22 in Book I.

It is useful to look at its organization.

- Definitions. Definitions in Book I are not formalized as in later texts. There is not much of interest here except that Euclid omitted about as many definitions as he gave.
- Problems. Problems are cast with an infinitive: "to find" or "to construct". A problem refers to the production of a specific figure from specific conditions. There are decided advantages to a problem approach [9, p.348–354]. Locus problems arise when there are multiple solutions.
- Postulates. There are three geometric postulates. These are stated in the form of *problems* and not *theorems*. That is, the postulates tell us that certain figures can be found given certain conditions and not merely that solutions "exist." The clear implication here is that the postulates were minimal with respect to solving problems: we need not concern ourselves with "simpler" problems. These postulates make for an ordering of problems [9, p. 350].

  Before continuing, let us be clear on this point: there are no axioms in Euclid and the postulates are not stated as relations but as algorithms!
- Common Notions. Common notions are what we would call axioms and are related to theorems and statements/proofs [7, p. 169]. It is not clear that the postulates as translated fit in this category or not. There appears to be quite a bit of doubt of as to what Euclid actually included where [5, 195–240]. We note also that the modern concept of axiom comes much later. In Euclid's time, common notions were made for verification (testing) purposes [6]. In view of the comment about postulates, axioms allow for the ordered sequence of theorems.
- Theorems. Theorems are cast as a conditional assertion relating to a specified configuration. They refer to general classes of objects. Theorems are the appropriate mode when one is concerned with formal organization of known results, rather than with the discovery of what is yet unknown [9]. For example, I.20 is a theorem, not a problem although it seems more naturally stated as a problem.

- Analysis. In analysis, we "pretend" to have the figure and reason backwards towards axioms or previous theorems [9, p. 354–360]. Aristotle holds that analysis is deductively admissible [9, p. 75].
- Synthesis. The desired figure is constructed by starting with the hypotheses and moving forward deductively to the conclusion [9, p. 354–360].
- Diorisms. Diorisms were formalized by the mid-4th century BC. Diorisms give auxiliary conditions which, when supplied, guarantee that a problem is solvable [9, p. 358]. Thus diorisms allow analysis to be converted to synthesis.

3.3. **The Methods of the Elements.** The standard method calls for analysis first, then synthesis [9, p. 9]. From Hero's comments on *Metrica* [9, fn 87, p. 376]: "The objective is to derive a computational rule or procedure by means of what is called an 'analysis' and then in the following 'synthesis' to work out a solution in particular numbers via the derived rule." The editorial style of the commentators seems to prefer the synthetic treatment but this obscures the essential lines of thought [9, p. 9].

The *Elements* is about nothing if it is not about justification.

> "In dialectics the grounds of knowledge become a central issue. What is true? How does one know what is true and distinguish from the false? How does one communicate and teach? .... But surely this general environment had an equivalent impact on mathematics. Examining their arithmetic and geometric techniques, they began to seek justifications .... Once the quest for justification was underway, the nature of mathematics itself would lead to deductive forms .... Furthermore, the incentive to teach spurred efforts to organize large areas of mathematics into coherent systems." [7, p. 179]

However, this sheds no light on what Euclid thought was sufficient justification. Surely Euclid would object to hiding the lines of development from his students: after all, as a teacher he would want his students to understand how the constructions and theorems come about. We turn to this in Section 4.

3.3.1. *The Theorems* versus *Problems Debate.* The historical record indicates that ancient commentators were often tainted by their own philosophical positions: Proclus is a good example. These predispositions obscure the debate over the primacy of theorems over problems. We seem to have more information on the philosophers' positions than the geometers' positions.

Proclus observed that Book I is evenly split between theorems and problems. "In this way, the problems often serve as justification for the introduction of auxiliary terms in later propositions" [9, p. 350]. Problems and theorems are interchangeable. Perhaps more important is the idea that successful solution to problems leads to theorems and *vice versa.* In my own experience, this relationship is crucial, as the insight needed to prove a theorem is often buried in the construction. Amphimonous, who is linked to Plato's successor and nephew Speusippus, is said to have held to the view that everything should be stated as a theorem [9, p. 75]. "Through all of this, *the technical literature does not suggest that everything should be in one form: theorem exclusive or problem.*" [Italics mine] [9, fn 60, p. 374].

3.3.2. *The Links Between The* Elements *and Philosophy.* Euclid's *Elements* and Aristotle's *Posterior Analytics* represent the state of the art at the time for geometry and philosophy respectively [7, p. 164–165]. "In this way, Aristotle's discussion of the first principles of proof becomes a straight-forward commentary on the actual procedures of proof employed by Euclid's immediate predecessors [7, p. 166–167]. "Indeed, judging from the epistemological views of Plato and Aristotle, one cannot escape the conviction that the influence of mathematics on philosophy was far more significant than any influence in the converse direction" [7, p. 179]. On the other hand, McKirahan seems not so sure [10].

4. A Possible Philosophy of Geometry

The historical study enables us to at least conjecture what the *geometers* thought about how the study of geometry should be conducted. A naive reading of the commentaries of Euclid would lead one to think that there was much philosophical input on the conduct of geometric inquiries. Eudoxus seems to be the only ancient geometer who was interested in formal considerations. More importantly,

> "Thus there appears to be no dialectical motive behind Euclid's statement of postulates or his presentation of geometric construction. Moreover, in setting out the postulates, he did not aim to restrict the whole field of research on constructions to those which can be effected in practice or even to suggest that these means were somehow more privileged among the variety of construction devices possible[example: cube duplication by non-Euclidean means]." [7]

Knorr points out that virtually any philosophical system, ancient or modern, is supported [8, p. 141] by the *Elements*. This means that one could take a philosophical point of view, like constructionism and claim that Euclid used that view to develop the *Elements*. This would seem to fit Proclus nicely. The commentators did help maintain information about "metageometry" issues by discussing certain issues. Therefore, it is likely that the geometers themselves had no commitment to philosophical positions in geometry. The technical treatises are devoid of philosophical concerns [8, p. 140–141].

4.1. **What Did Geometers Think About.** If not driven by philosophical ideas, what did the geometers concern themselves with in their investigations? A short list:

A. How should the field be partitioned by problem and solution methodology?

B. What is the role of problems *vis-à-vis* theorems?

C. What is the status of analysis *vis-à-vis* synthesis?

D. What conditions should be imposed on which allowable technologies?

E. What are the judgments for accepting a solution that has been found to a problem?

It should be noted that almost exactly these same questions are being asked in the reformation of calculus.

For our purposes, a small number of questions need to be addressed in detail.

4.1.1. *Superposition.* A fundamental approach to the computable theory rests on superposition. The formalist commentators hold that Euclid used it reluctantly. Knorr investigates superposition in detail [7, p. 159–161] and concludes that Euclid probably did not shun the concept but rather found it was not needed because of the order of presentations.

4.1.2. *Axiomatization.* While axioms play a major role in mathematics today, they seemed not to be so important in ancient times. I have made a case that the technical treatises do not delve into subtle questions of axiomatics [9, p. 8]. But what were the ancient mathematical views?

A. "The notion that ancient mathematics was somehow a vast exercise in dialectical philosophy must miss a very important point: that geometry is rooted in an essentially practical enterprise of problem solving." [9, p. 11]

B. "The subsequent history of mathematics indicates that the success of axiomatizing effort [by the 3rd century] eventually served to discourage the creative forms of research which could have advanced mathematical knowledge." [7, p. 178]

To develop the constructionist viewpoint, we replace the concept that axioms are logical statements expressing fundamental conditions of the system by primitive computational capabilities that achieve the same goal. Once the computational capabilities are associated with simple relations we can expand the relations into logical statements.

> "By emphasizing the solution of problems the ancients do not necessarily intend physical constructions, although the literature includes many examples of the type. But neither

> do they emphasize the pure aspects of theory to the exclusion of interests in the practical aspects of their results. Even in these advanced theoretical efforts in geometry, the ancients are still sensitive, if only in an indirect way, to the demands of practice. [8, p. 140]"

Thus, the ancients would probably accept Brouwer's view: mathematics is done with mental constructions. The ancient geometers would basically approve of a constructive formulation that emphasizes *idealization* over *axiomatization.*

4.2. **Restrictions to Ruler and Compasses.** Oenipides is blamed for this restriction, but I have already made a case against the view in the section above. The restriction to ruler and compass, then, should be seen as a pedagogical one. It is clear that much of geometry depends on the ruler and the compass. This might lead a philosophically oriented ancient geometer to formulate a version of Church's Thesis: Anything that is Euclidean is constructible by a ruler and compass. We might ask, for example, if there are simpler procedures (for ruler only, say) which produce sub-theories? As we well know, the answer is yes [3]. In light of modern studies, the restriction is a challenge to action, not a prohibition. We also know that ruler and compasses are not enough to solve all problems posable in geometry, leaving open the question of what is an adequate base set of procedures.

4.3. **The Balance between Practical and Theoretical.** There has been a tendency in recent years to see mathematics as detached from its application. You might call this the *pure versus applied* debate. In my opinion, this is not even a sensible question, since elements of mathematics (and other sciences as well) can serve in both capacities; clearly, *pure* drives *applied* and *vice versa.*

The history of mathematics shows that formal structure can slow down the search for solutions but sometimes demand great subject insight [9, p. 76]. While we give great credit to Euclid for presenting formalized methods, it is not until Appolonius that the subject matter gets formalized into formal methods of problem solving [9, p. 346].

Mathematics and science bounce between two extremes: On the one hand we have, "The modern mathematician may be remote from such practical applications of his researches, but this is clearly not the case with the ancients. [8, p. 139]". The 20th century shows "developments in analysis, geometry, number theory, and the theory of sets revealed that certain formal issues, long ignored or unperceived lay at the heart of important technical difficulties." [9, p. 87]

4.4. **What Do We Conclude?**

"To every thing there is a season, and a time to every purpose under heaven" *Ecclesiastes* 3:1

What would ancient researchers say if we asked, "What are the principles that guide the investigation of geometry?" I think they might answer something like this:

> "We study and solve problems in a general sense. There are problems that we can solve with our current methods and there are problems we cannot solve—or at least don't know how to solve with the current methods. We use problems to motivate our research. There are general theorems about geometry but we have no catalog of all knowable facts about what you call Euclidean geometry.
>
> "As to formalization, we have no prescribed set of concepts nor do we attempt to ban concepts. Problems are approached by one set of techniques, and theorems by another, but they seem to be inter-related. There is some discussion as to how these are inter-related by analysis and synthesis. The ultimate question is, 'How do we know we are right?' Problems are the life blood of any work."
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> "To paraphrase Wilbur Knorr, 'You assume that the hallmark of our work is our organization, tight structures of deductive reasoning and fully justified problem solutions' [9, p. 7]. No, we're just like you .... We put it down and refine and argue over the correctness of our thoughts. We have our good days and our bad days."

5. What Does This Have To Do With Computer Science

The explosive growth of computer technology and the number of people "practicing" computer science is a historical first. The Information Age and the Information Revolution have spawned tremendous changes in science, mathematics, and a host of sociological process in business, warfare, and home life. We will not be in a position to take twenty-three centuries to develop computer science in the "slow" way mathematics developed. But we can hope that computer science is as good an acorn as geometry has been for the oak tree of mathematics. Euclid does speak to computer science, though.

5.1. Some Observations.

A. It is clear that practical problems are interesting. The "Traveling Salesperson Problem" is such an example. What is allowable technology (compass and ruler only argument) has not been a problem because the chip manufacturers develop more and faster chips.

B. Forcing artificial restrictions on method is to be avoided. Perhaps the ancients had an unfair advantage: they invented the academic system but didn't have to live in it. One artificial restriction is that computation is exactly an axiomatic structure. Fetzer has convincingly countered this argument in [4], to the howls of the computer science establishment.

   Another aspect of this problem of constraints has been to demand that computer science follow mathematical epistemology of proof by derivation along mathematical logical structures. I characterize this problem as the "Hammer Problem." A long standing joke in engineering says, "To a engineer with a hammer the whole world looks like a nail." To engineers, mathematicians, and scientists with their hammers, computer science looks like a nail.

C. The interchangeability of problems and theorems is an interesting extension. An intriguing aspect is how to link them formally — a constructivist's goal.

D. Formalizing the analysis and synthesis aspects of computing would be an important step towards effective teaching of concepts and generating generations of competent computer scientists. By the way, I have not one clue as to how to translate the concept of *diorisms*.

5.2. **Comments on Method.** My original goal was to understand Euclid in terms of his method with a "Hammer" of constructivism not Platonism. What can I say about that?

Because the postulates are stated as problems, I can at least conclude that the idea of construction is prior to theorem. This is not completely surprising since systems of axioms must be verified for consistency. Also interesting is the observation that the reasoning rules were for verification, not derivation.

From Hero's comments in Section 3.3, we can conclude something about how "methods" were viewed. The goal was to derived a *computational rule* that linked analysis to synthesis. In this, apparently, the ancients say synthesis as the goal — just like programming.

To close this section, geometry fairly well shouts advice about formalization: "The subsequent history of mathematics indicates that the success of axiomatizing effort [by the 3rd century] eventually served to discourage the creative forms of research which could have advanced mathematical knowledge" [7, p. 178]. Computer science is full of efforts to over-formalize everything. We seem not to have learned much through history.

5.3. **So What?** I would propose that a goal of academicians should be to come up with *Euclid's Elements* for computer science: a catalog of all the methods and problems that we can solve. Teaching computer science might far more rewarding and less magical if we had such a text. After all, it nourished twenty-three hundred years of mathematicians.

It is a pity we no longer teach it in our schools and computer scientists are ignorant of its legacy.

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Lastly, as I read Professor Knorr's texts, I became increasing sorry I never knew him. I hope I've been a good interpreter of the evidence.

References

[1] Errett Bishop and Douglas Bridges. *Constructive Analysis*. Springer-Verlag, 1985.

[2] Carl B. Boyer. *A History of Mathematics*. Wiley, 1989.

[3] H.S.M. Coxeter. *Introduction to Geometry*. John Wiley and Sons, Inc., New York, 2 edition, 1969.

[4] James H. Fetzer. Program verification: the very idea. *CACM*, 31(9), 1988.

[5] Sir Thomas L. Heath. *The Thirteen Books of Euclid's Elements*, volume I–III. Dover Publications, Inc., New York, 1956.

[6] Morris Kline. *Mathematical Thought from Ancient to Modern Times*. Oxford University Press, 1972.

[7] Wilbur R. Knorr. Infinity and continuity: The interaction of mathematics and philosophy in antiquity. In Norman Ketzmann, editor, *Infinity and Continuity in Ancient and Medieval Thought*, pages 146–164. Cornell University Press, 1981.

[8] Wilbur R. Knorr. Construction as existence proof in ancient geometry. *Ancient Philosophy*, pages 125–148, 1983.

[9] Wilbur R. Knorr. *The Ancient Tradition of Geometric Problems*. Birkhäser, 1986.

[10] Richard D. McKirahan. *Principles and Proofs: Aristotle's Theory of Demonstrative Science*. Princeton University Press, 1992.

[11] A. Seidenberg. *Peg and Cord in Ancient Greek Geometry*, volume 24. Scripta Mathematica, 1959.

[12] A. Seidenberg. The ritual origin of geometry. *History of the Exact Sciences*, 1:487–527, 1963.

[13] G. L. Toomer. Euclid. In *Oxford Classical Dictionary*, page 564. Oxford University Press, 1996.

[14] Walter P. van Stigt. *Brouwer's Intuitionism*, chapter 2–4. Elsevier Science Publishing Company, Inc., 1990. QA29.B697S25.

[15] Jan von Plato. The axioms of constructive geometry. *Annals of Pure and Applied Logic*, 76:169–200, 1995.

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